THE GROWTH SEQUENCE OF SYMPLECTOMORPHISMS ON SYMPLECTICALLY HYPERBOLIC MANIFOLDS

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Abstract. We study the growth rate of a sequence which measures the uniform norm of the differential under the iterates of maps. On symplectically hyperbolic manifolds, we show that this sequence has at least linear growth for every non-identical symplectomorphisms which are symplectically isotopic to the identity.

1. Introduction

Let (M, g) be a closed connected Riemannian manifold. Given a diffeomorphism φ on M, its *growth sequence* is defined by

$$\Gamma_n(\varphi) := \max\left(\max_{m\in M}|d\varphi^n(m)|_g, \max_{m\in M}|d\varphi^{-n}(m)|_g\right),\ n\in\mathbb{N}.$$

Here the norm of the differential is given by the operator norm. The explicit value of the growth sequence is hard to compute, with the following terminology however, we measure the growth rate of $\Gamma_n(\varphi)$. Given two positive sequences $a_n, b_n : \mathbb{N}\to[0,\infty)$, let us denote $a_n \lesssim b_n$ if there exists $c > 0$ such that $a_n \le c(b_n + 1)$ for all $n \in \mathbb{N}$.

We investigate $\Gamma_n(\varphi)$ for a symplectomorphism φ in the identity component of the group of symplectomorphism $Symp_0(M,\omega)$ of a symplectic manifold (M,ω). Polterovich proved, in his beautiful paper [Pol], many interesting results about the growth type on $\Gamma_n(\varphi)$ for $\varphi \in Symp_0(M,\omega)\setminus\{\mathbb{1}\}$ on a closed symplectic manifold (M,ω) with $\pi_2(M) = 0$. Among them, let us state a related result to this article as follows: A closed symplectic manifold (M,ω) with $\pi_2(M) = 0$ is called *symplectically hyperbolic*, if the symplectic form ω admits a bounded primitive on the universal cover $p : \widetilde{M} \to M$. On this symplectically hyperbolic manifold (M,ω), Polterovich showed $\Gamma_n(\varphi) \gtrsim n$ when $\varphi \in Symp_0(M,\omega)\setminus\{\mathbb{1}\}$ has a fixed point of contractible type.

The existence of fixed points with positive action difference is crucial in his proof. When φ is a Hamiltonian diffeomorphism, the existence of a fixed point is obtained from Floer's proof [Fl] of Arnold's conjecture and another fixed point with different action is established in the work of Schwarz [Sch] by using Floer homology. For a non-Hamiltonian diffeomorphism, we have a non-trivial flux, see Definition 2.7. This guarantees another fixed point with a different action from the given fixed point.

In 2-dimensional case i.e. a closed oriented surface Σ_g of genus $g \ge 2$, see Example 2.4, we know that there exists a fixed point of $\varphi \in Symp_0(\Sigma_g,\omega)$. In [Gro1], Gromov proved that $(-1)^n\chi(M) > 0$ for a symplectically hyperbolic manifold (M,ω) which is Kähler.Using this non-vanishing Euler characteristic, one can also obtain a fixed point of $\varphi \in Symp_0(M,\omega)$ as in [Pol, Example 1.3.C]. In the above cases, we hence conclude $\Gamma_n(\varphi) \gtrsim n$ for $\varphi \in$

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$Symp_0(M,\omega) \setminus \{\mathbb{1}\}$, but there is no known result about the existence of fixed points of φ on symplectically hyperbolic manifolds in general. We will prove the following statement even though there is no assumption on a fixed point.

Main Theorem. *Let (M,ω) be a symplectically hyperbolic manifold. If $\varphi \in Symp_0(M,\omega) \setminus \{\mathbb{1}\}$, then $\Gamma_n(\varphi) \gtrsim n$.*

Our main tools are the mapping torus construction and the cofilling function of the twisted Hamiltonian structure which will be introduced in the following section. We interpret the non-trivial flux of non-Hamiltonian diffeomorphism into a certain type of cofilling function.

The concept of the growth sequence can be generalized as follows: Let Σ_i be the set of smooth embeddings σ of an i-dimensional cube $[0,1]^i$ into M. Denote by $\mu(\sigma)$ the volume of $\sigma([0,1]^i) \subset M$ with respect to g. For each $i \in \{1,\dots,\dim M\}$, we define the *i-th slow volume growth* by

$$s_i(\varphi) := \sup_{\sigma\in\Sigma_i} \liminf_{n\to\infty} \frac{\log \mu(\varphi^n(\sigma))}{\log n}.$$

If φ is compactly supported, $s_i(\varphi)$ is independent of the choice of g. One can check that $\Gamma_n(\varphi) \gtrsim n$ if and only if $s_1(\varphi) \geq 1$. These entropy-type invariants measure the orbit structure complexity of a smooth diffeomorphism φ, see [KH] for the related topics.

Corollary. *Let (M,ω) be a symplectically hyperbolic manifold. If $\varphi \in Symp_0(M,\omega) \setminus \{\mathbb{1}\}$, then $s_1(\varphi) \geq 1$.*

In [FS1, FS2, FS3], Frauenfelder and Schlenk considered $s_1(\varphi)$ for $\varphi \in Symp_0^c(T^*B, d\lambda)\setminus\{\mathbb{1}\}$ and $s_{\dim B}(\varphi)$ for $\varphi \in Symp_0^c(T^*B, d\lambda) \setminus \{\mathbb{1}\}$ of the Dehn-Seidel twist type on a compact rank one symmetric space B. They interpreted Lagrangian intersections into the slow volume growth of Lagrangian submanifolds by using Lagrangian Floer homology.

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2. Preliminaries

2.1. Cofilling function.

Definition 2.1 ([Gro2], [Pol]). Let $\sigma \in \Omega^2(M)$ be a closed 2-form and $\widetilde{\sigma} \in \Omega^2(\widetilde{M})$ is exact. Let $\mathcal{P}_\sigma$ be the space of all 1-forms on $\widetilde{M}$ whose differential is $\widetilde{\sigma}$. Pick a point $x \in \widetilde{M}$ and denote by $B_x(s)$ the ball of radius $s > 0$ with respect to $\widetilde{g}$ which centered at x. Then the *cofilling function* $u_\sigma : [0,\infty) \to [0,\infty)$ is defined by

$$u_\sigma(s) = u_{\sigma,g,x}(s) := \inf_{\theta\in\mathcal{P}_\sigma} \sup_{z\in B_x(s)} |\theta_z|_{\widetilde{g}}.$$

Here the norm of a differential form is

$$|\theta_z|_{\widetilde{g}} := \max\{\theta_z(v) : \|v\|_{\widetilde{g}} = 1\}.$$

Given two functions $f, g : [0,\infty) \to [0,\infty)$, we write $f \lesssim g$ if there exists $c > 0$ such that $f(s) \leq c(g(s)+1)$ for all $s \in [0,\infty)$, and $f \sim g$ if $f \lesssim g$, $g \lesssim f$. If we choose

another Riemannian metric g' on M and a different base point x' then we can check that $u_{\sigma,g,x} \sim u_{\sigma,g',x'}$. This means that the growth type of the cofilling function is an invariant of the closed 2-form σ.

Proposition 2.2. *The growth type of the cofilling function only depends on the cohomology class.*

PROOF. Let σ, σ' be closed 2-forms on a closed manifold M and $\widetilde{\sigma}$, $\widetilde{\sigma}'$ are exact. Assume $[\sigma] = [\sigma']$, then we need to show that $u_\sigma(s) \sim u_{\sigma'}(s)$. Since σ' is cohomologous to σ there exists a 1-form ξ such that $\sigma' = \sigma + d\xi$. Now we compute

$$\begin{aligned}
u_{\sigma'}(s) &= \inf_{\theta\in\mathcal{P}_{\sigma'}} \sup_{z\in B_x(s)} |\theta_z|_{\widetilde{g}} \\
&= \inf_{\theta\in\mathcal{P}_{\sigma}} \sup_{z\in B_x(s)} |(\theta+\widetilde{\xi})_z|_{\widetilde{g}} \\
&\leq \inf_{\theta\in\mathcal{P}_{\sigma}} \sup_{z\in B_x(s)} |\theta_z|_{\widetilde{g}} + \sup_{z\in B_x(s)} |\widetilde{\xi}_z|_{\widetilde{g}} \\
&\leq \inf_{\theta\in\mathcal{P}_{\sigma}} \sup_{z\in B_x(s)} |\theta_z|_{\widetilde{g}} + \max_{z\in M} |\xi_z|_g \\
&= u_\sigma(s) + C,
\end{aligned}$$

where $C = \max_{z\in M} |\xi_z|_g$. In a similar way we obtain $u_\sigma(s) \leq u_{\sigma'}(s) + C$. These conclude $u_\sigma(s) \sim u_{\sigma'}(s)$. $\square$

Example 2.3. Consider the standard symplectic torus ($\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, $\omega = \Sigma_{i=1}^n dx_i \wedge dy_i$) with the metric induced by the Euclidean metric on $\mathbb{R}^{2n}$. Since $\widetilde{\omega} = d(\Sigma_{i=1}^n x_i \wedge dy_i) \in \Omega^2(\mathbb{R}^{2n})$ and $\Sigma_{i=1}^n x_i \wedge dy_i$ has linear growth with respect to the Euclidean metric on $\mathbb{R}^{2n}$, it follows that $u_\omega(s) \lesssim s$.

For any primitive $\alpha \in \Omega^1(\mathbb{R}^{2n})$ of $\widetilde{\omega}$, we obtain

$$\frac{\pi^n}{n!}s^{2n} = \int_{B_x(s)} \widetilde{\omega} = \int_{\partial B_x(s)} \alpha \leq \sup_{z\in\partial B_x(s)} |\alpha_z| \cdot \int_{\partial B_x(s)} 1 \leq \sup_{z\in\partial B_x(s)} |\alpha_z| \cdot \frac{2\pi^n}{(n-1)!}s^{2n-1}.$$

This implies $\sup_{z\in B_x(s)} |\alpha_z| \geq \frac{s}{2n}$ and hence we conclude that $u_\omega(s) \sim s$.

Example 2.4. Let (Σ_g, ω) be a closed oriented surface of genus $g \geq 2$ with a volume form. First represent M as $\mathbb{H}/G$, where $\mathbb{H} = \{x + yi \in \mathbb{C} : y > 0\}$ is the hyperbolic upper half-plane and G is a discrete group of isometries. Without loss of generality, we may assume that the lift $\widetilde{\omega}$ of ω to the universal cover $\mathbb{H}$ coincide with the hyperbolic area form $\frac{1}{y^2}dx \wedge dy$. Note that $\widetilde{\omega} = d\big(\frac{1}{y}dx\big)$ and $\frac{1}{y}dx$ is bounded with respect to the hyperbolic metric $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$ on $\mathbb{H}$. By the definition, (Σ_g, ω) is a symplectically hyperbolic manifold and every symplectically hyperbolic manifold satisfies $u_\omega(s) \sim 1$.

Proposition 2.5. *Let M, N be closed manifolds and $h : N \to M$ be an immersion. Let $\sigma \in \Omega^2(M)$ be a closed 2-form which is exact on $\widetilde{M}$, then $u_{h^*\sigma}(s) \lesssim u_\sigma(s)$.*

Proof. Let g be a Riemannian metric on M then h^*g gives a Riemannian metric on N. We have the following commutative diagram on the universal covers:

$$\begin{tikzcd} (\widetilde{N}, \widetilde{h}^*\widetilde{g}) \arrow[r, "\widetilde{h}"] \arrow[d, "p_N"] & (\widetilde{M}, \widetilde{g}) \arrow[d, "p_M"] \\ (N, h^*g) \arrow[r, "h"] & (M, g) \end{tikzcd}$$

where $\widetilde{[-]}$ are the lifts of $[-]$ to the universal covers. Now choose a primitive $\theta \in \Omega^1(\widetilde{M})$ of $\widetilde{\sigma}$, then $\widetilde{h}^*\theta \in \Omega^1(\widetilde{N})$ is a primitive of $\widetilde{h}^*\widetilde{\sigma}$. Pick a point $z \in \widetilde{N}$, we note that

$$\begin{aligned} |(\widetilde{h}^*\theta)_z|_{\widetilde{h}^*\widetilde{g}} &= \max\{(\widetilde{h}^*\theta)_z[v] : \|v\|_{\widetilde{h}^*\widetilde{g}} = 1,\, v \in T_z\widetilde{N}\} \\ &= \max\{\theta_{\widetilde{h}(z)}[\widetilde{h}_*v] : \|\widetilde{h}_*v\|_{\widetilde{g}} = 1,\, v \in T_z\widetilde{N}\} \\ &\leq \max\{\theta_{\widetilde{h}(z)}(v) : \|v\|_{\widetilde{g}} = 1,\, v \in T_{\widetilde{h}(z)}\widetilde{M}\} \\ &= |\theta_{\widetilde{h}(z)}|_{\widetilde{g}}. \end{aligned}$$

This implies the following computation:

$$\begin{aligned} u_{h^*\sigma}(s) &\sim \inf_{\theta \in \mathcal{P}_{h^*\sigma}} \sup_{z \in B_{\widetilde{N},x}(s)} |\theta_z|_{\widetilde{h}^*\widetilde{g}} \\ &\lesssim \inf_{\theta \in \mathcal{P}_\sigma} \sup_{z \in B_{\widetilde{N},x}(s)} |(\widetilde{h}^*\theta)_z|_{\widetilde{h}^*\widetilde{g}} \\ &\lesssim \inf_{\theta \in \mathcal{P}_\sigma} \sup_{z \in B_{\widetilde{N},x}(s)} |\theta_{\widetilde{h}(z)}|_{\widetilde{g}} \\ &\lesssim \inf_{\theta \in \mathcal{P}_\sigma} \sup_{z \in B_{\widetilde{M},\widetilde{h}(x)}(s)} |\theta_z|_{\widetilde{g}} \\ &\sim u_\sigma(s), \end{aligned}$$

which concludes the proof.

□

Corollary 2.6. *If (M, ω) is a symplectically hyperbolic manifold then*

$$\int_{f(\mathbb{T}^2)} \omega = 0$$

for any immersion $f : \mathbb{T}^2 \to M$.

Proof. If the integral is non-zero, then $[f^*\omega] \neq 0$ in $H^2(\mathbb{T}^2, \mathbb{R})$. By Example 2.3 and Proposition 2.5, we deduce the following contradiction:

$$s \sim u_{f^*\omega}(s) \lesssim u_\omega(s) \sim 1.$$

□

Indeed, Corollary 2.6 holds true for any smooth map $f : \mathbb{T}^2 \to M$ and the proof is almost same, see [Kęd].

2.2. **Flux homomorphism.** Let $\varphi \in Symp_0(M,\omega)$ and $\{\varphi_t\}_{t\in[0,1]}$ a path of symplectomorphisms such that $\varphi_0 = \mathbb{1}$ and $\varphi_1 = \varphi$. We define an induced vector field Y_t by

$$\frac{d}{dt}\varphi_t = Y_t \circ \varphi_t \tag{2.1}$$

which is called a *symplectic vector field.* Since the Lie derivative $\mathcal{L}_{Y_t}\omega$ vanishes, we get a closed 1-form $\iota_{Y_t}\omega$ for each $t \in [0,1]$.

Definition 2.7. The *flux homomorphism* $\widetilde{Flux} : \widetilde{Symp}_0(M,\omega) \to H^1(M,\mathbb{R})$ is defined as follows:

$$\widetilde{Flux}\big(\{\varphi_t\}\big) = \int_0^1 [\iota_{Y_t}\omega]\, dt,$$

where $[\alpha]$ is the cohomology class of a form α.

The kernel of $\widetilde{Symp}_0(M,\omega) \to Symp_0(M,\omega)$ can be identified with the fundamental group $\pi_1(Symp_0(M,\omega))$. We denote by $\Gamma_\omega \subset H^1(M,\mathbb{R})$ the image of $\pi_1(Symp_0(M,\omega))$ by $\widetilde{Flux}$ and we call Γ_ω the *flux group* of (M,ω). Then $\widetilde{Flux}$ descends to a homomorphism

$$Flux \colon Symp_0(M,\omega) \to H^1(M,\mathbb{R})/\Gamma_\omega.$$

The next well-known fact is proved in [MS].

Theorem 2.8. *$Ham(M,\omega) = \ker Flux$.*

Proposition 2.9 ([Kęd], [Pol])**.** *Let (M,ω) be a symplectically hyperbolic manifold, then the flux group $\Gamma_\omega = 0$ in $H^1(M,\mathbb{R})$.*

The *atoroidal* property of ω, $\int_{f(\mathbb{T}^2)} \omega = 0$ for any smooth $f : \mathbb{T}^2 \to M$, is important in the proof of Proposition 2.9. We then have

$$Flux \colon Symp_0(M,\omega) \to H^1(M,\mathbb{R}).$$

2.3. **Hamiltonian structures.** Let Σ be a closed connected orientable manifold of dimension $2n+1$. A *Hamiltonian structure* on Σ is a closed 2-form ω such that ω^n is nowhere vanishing. So its kernel $\ker\omega$ defines a 1-dimensional foliation which we call the *characteristic foliation* of ω.

Definition 2.10. A Hamiltonian structure (Σ,ω) is called *stable*, if there exists a 1-form λ such that

$$\ker\omega \subset \ker d\lambda, \qquad \lambda\wedge\omega^n > 0. \tag{2.2}$$

We call the 1-form λ a *stabilizing 1-form.* This structure defines the *Reeb vector field R* by

$$\lambda(R) = 1, \qquad \iota_R\omega = 0.$$

Definition 2.11. A Hamiltonian structure is called *virtually contact*, if there is a covering $p : \widehat{\Sigma} \to \Sigma$ and a primitive $\lambda \in \Omega^1(\widehat{\Sigma})$ of $p^*\omega$ such that

$$\sup_{x\in\widehat{\Sigma}} |\lambda_x| \le C < \infty, \qquad \inf_{x\in\widehat{\Sigma}} \lambda(R) \ge \mu > 0,$$

where $|\cdot|$ is the lifting of a metric on Σ and R is the pullback of a unit vector field generating $\ker\omega$.

3. Hamiltonian structures on mapping tori

For a symplectically hyperbolic manifold (M,ω) and a symplectomorphism $\varphi \in Symp_0(M,\omega)$, we consider a *mapping torus* M_φ *of* M *with respect to* φ which is constructed as follows:

$$M_\varphi = \frac{M\times[0,1]}{(m,0)\sim(\varphi(m),1)}. \tag{3.1}$$

Now we consider two mapping tori $M_{\mathbf{1}}$, M_φ and Hamiltonian structures on the tori. The trivial mapping torus $M_{\mathbf{1}} \cong M\times\mathbb{S}^1$ carries a Hamiltonian structure $\omega_{\mathbf{1}} := \pi^*\omega$, where $\pi : M_{\mathbf{1}} \to M : (m,\theta)\mapsto m$. The non-trivial one M_φ admits a Hamiltonian structures $\omega_\varphi := \pi_\varphi^*\omega$, where $\pi_\varphi : M_\varphi \to M : (m,\theta)\mapsto m$. The Hamiltonian structure ω_φ is a well-defined 2-form on M_φ, since $\varphi^*\omega = \omega$. Note that the both Hamiltonian structures have the kernel which is spanned by $\frac{\partial}{\partial\theta}$. Let us choose a path of symplectomorphism $\{\varphi_t\}_{t\in[0,1]}$ from $\varphi_0 = \mathbb{1}$ to $\varphi_1 = \varphi$. The twisting map $f : M_{\mathbf{1}} \to M_\varphi$ is then given by

$$f(m,\theta) = (\varphi_\theta(m),\theta). \tag{3.2}$$

Lemma 3.1. Let $(M_{\mathbf{1}},\omega_{\mathbf{1}})$, $(M_\varphi,\omega_\varphi)$ and $f : M_{\mathbf{1}} \to M_\varphi$ be as above, then we obtain

$$f^*\omega_\varphi = \omega_{\mathbf{1}} - \pi^*(\iota_{Y_\theta}\omega)\wedge\pi_\theta^* d\theta, \tag{3.3}$$

where $\pi_\theta : M_{\mathbf{1}} \to \mathbb{S}^1 : (m,\theta)\mapsto\theta$.

Proof. Let (m_i,θ_i) be a tangent vector in $T_{(m,\theta)}M_{\mathbf{1}}$. Note that we identify $T_\theta\mathbb{S}^1$ with $\mathbb{R}$, so θ_i is considered as an element of $\mathbb{R}$. We then consider $f^*\omega_\varphi$ as follows:

$$\begin{aligned}
(f^*\omega_\varphi)_x\big((m_1,\theta_1),(m_2,\theta_2)\big) &= (\omega_\varphi)_{f(x)}\big(f_*(m_1,\theta_1), f_*(m_2,\theta_2)\big)\\
&=(\omega_\varphi)_{f(x)}\big((d\varphi_\theta(m)[m_1]+\theta_1\cdot Y_\theta[\varphi_\theta(m)],\theta_1),(d\varphi_\theta(m)[m_2]+\theta_2\cdot Y_\theta[\varphi_\theta(m)],\theta_2)\big)\\
&=(\omega_\varphi)_{f(x)}\big((\theta_1\cdot Y_\theta[\varphi_\theta(m)],0),(d\varphi_\theta(m)[m_2],\theta_2)\big)\\
&\quad+(\omega_\varphi)_{f(x)}\big((d\varphi_\theta(m)[m_1],\theta_1),(\theta_2\cdot Y_\theta[\varphi_\theta(m)],0)\big)\\
&\quad+(\omega_\varphi)_{f(x)}\big((\theta_1\cdot Y_\theta[\varphi_\theta(m)],0),(\theta_2\cdot Y_\theta[\varphi_\theta(m)],0)\big)\\
&\quad+\underbrace{(\omega_\varphi)_{f(x)}\big((d\varphi_\theta(m)[m_1],\theta_1),(d\varphi_\theta(m)[m_2],\theta_2)\big)}_{=:\diamond}.
\end{aligned}$$

Here Y_θ is the symplectic vector field defined in (2.1). In order to simplify the $\diamond$-term we compute

$$\begin{aligned}
\diamond &=(\pi_\varphi^*\omega)_{f(x)}\big((d\varphi_\theta(m)[m_1],\theta_1),(d\varphi_\theta(m)[m_2],\theta_2)\big)\\
&=\omega_{\varphi_\theta(m)}(d\varphi_\theta(m)[m_1],d\varphi_\theta(m)[m_2])\\
&=(\varphi_\theta^*\omega)_m(m_1,m_2)\\
&=\omega_m(m_1,m_2),
\end{aligned}\tag{3.4}$$

where the last equality comes from the assumption that $\varphi_\theta : M\to M$ is a symplectomorphism. We also know

$$\begin{aligned}
(\omega_{\mathbf{1}})_x\big((m_1,\theta_1),(m_2,\theta_2)\big) &=(\pi^*\omega)_x\big((m_1,\theta_1),(m_2,\theta_2)\big)\\
&=\omega_m(m_1,m_2).
\end{aligned}\tag{3.5}$$

By combining (3.4), (3.5) we obtain

$$(\omega_{\mathbf{1}})_x\big((m_1,\theta_1),(m_2,\theta_2)\big) = (\omega_\varphi)_{f(x)}\big((d\varphi_\theta(m)[m_1],\theta_1),(d\varphi_\theta(m)[m_2],\theta_2)\big). \tag{3.6}$$

Then the difference between $f^*\omega_\varphi$ and $\omega_{\mathbf{1}}$ becomes

$$\begin{aligned}(f^*\omega_\varphi - \omega_{\mathbf{1}})_x\big((m_1,\theta_1),(m_2,\theta_2)\big) =&(\omega_\varphi)_{f(x)}\big((\theta_1\cdot Y_\theta[\varphi_\theta(m)],0),(d\varphi_\theta(m)[m_2],\theta_2)\big)\\ &+(\omega_\varphi)_{f(x)}\big((d\varphi_\theta(m)[m_1],\theta_1),(\theta_2\cdot Y_\theta[\varphi_\theta(m)],0)\big)\\ =&\omega_{\varphi_\theta(m)}\big(d\varphi_\theta(m)[m_1],\theta_2\cdot Y_\theta[\varphi_\theta(m)]\big)\\ &+\omega_{\varphi_\theta(m)}\big(\theta_1\cdot Y_\theta[\varphi_\theta(m)],d\varphi_\theta(m)[m_2]\big).\end{aligned} \tag{3.7}$$

On the other hand, we have

$$\begin{aligned}&\big(\pi^*(\iota_{Y_\theta}\omega)\wedge\pi_\theta^* d\theta\big)_x\big((m_1,\theta_1),(m_2,\theta_2)\big)\\ &=\big(\pi^*(\iota_{Y_\theta}\omega)\wedge\pi_\theta^* d\theta\big)_{f(x)}\big((d\varphi_\theta(m)[m_1],\theta_1),(d\varphi_\theta(m)[m_2],\theta_2)\big)\\ &=(\iota_{Y_\theta}\omega)_{\varphi_\theta(m)}\big(d\varphi_\theta(m)[m_1]\big)\cdot(d\theta)_\theta(\theta_2)\\ &\quad-(\iota_{Y_\theta}\omega)_{\varphi_\theta(m)}\big(d\varphi_\theta(m)[m_2]\big)\cdot(d\theta)_\theta(\theta_1)\\ &=\omega_{\varphi_\theta(m)}\big(Y_\theta[\varphi_\theta(m)],d\varphi_\theta(m)[m_1]\big)\cdot\theta_2\\ &\quad-\omega_{\varphi_\theta(m)}\big(Y_\theta[\varphi_\theta(m)],d\varphi_\theta(m)[m_2]\big)\cdot\theta_1.\end{aligned} \tag{3.8}$$

By combining (3.7) and (3.8), we conclude (3.3). This proves the lemma. □

Remark 3.2. The Hamiltonian structure ω_φ on M_φ has its kernel spanned by $\frac{\partial}{\partial\theta}$. If there exist a closed orbit $\gamma:\mathbb{S}^1\to M_\varphi$ of $\frac{\partial}{\partial\theta}$, then its projection $\pi\circ\gamma:\mathbb{S}^1\to M$ gives us a symplectic fixed point with respect to $\varphi\in Symp_0(M,\omega)$. One can easily check that $(f^*\omega_\varphi)(\frac{\partial}{\partial\theta}-Y_\theta)=0$. This implies that the vector field $\frac{\partial}{\partial\theta}-Y_\theta$ spans the $\ker f^*\omega_\varphi$ on $M_{\mathbf{1}}$. The closed orbit of $\frac{\partial}{\partial\theta}-Y_\theta$ also can be interpreted as a fixed point of the vector field of Y_θ.

Proposition 3.3. *Let (M,ω) be a symplectically hyperbolic manifold and $\varphi\in Symp_0(M,\omega)$. If $Flux(\varphi)\neq 0$ then the Hamiltonian structure $f^*\omega_\varphi\in\Omega^2(M_{\mathbf{1}})$ satisfies $u_{f^*\omega_\varphi}(s)\gtrsim s$.*

Proof. First recall that

$$f^*\omega_\varphi=\omega_{\mathbf{1}}-\pi^*(\iota_{Y_\theta}\omega)\wedge\pi_\theta^* d\theta.$$

Since our symplectic manifold (M,ω) is symplectically hyperbolic, the standard Hamiltonian structure $\omega_{\mathbf{1}}=\pi^*\omega\in\Omega^2(M_{\mathbf{1}})$ admits a bounded primitive $\widetilde{\pi}^*\lambda$ on $\widetilde{M}\times\mathbb{R}$, where $\widetilde{\omega}=d\lambda$ and $\widetilde{\pi}:\widetilde{M}\times\mathbb{R}\to\widetilde{M}$ is the lift of π.

Now we consider the twisted term $\pi^*(\iota_{Y_\theta}\omega)\wedge\pi_\theta^* d\theta$. Since $Flux(\varphi)$ is nontrivial in $H^1(M,\mathbb{R})$, there exists $a\in\pi_1(M)$ such that $\langle Flux(\varphi),\overline{a}\rangle\neq 0$, where $\overline{a}$ stands for the image of a in $H_1(M,\mathbb{Z})$ under the Hurewicz homomorphism. Without loss of generality, we may choose an immersed curve $\gamma:\mathbb{S}^1\to M$ such that $[\gamma]=a$.

Let us consider the induced immersion of $\mathbb{T}^2$

$$\begin{aligned}h:\mathbb{T}^2&\to M_{\mathbf{1}}=M\times\mathbb{S}^1\\ (t,\theta)&\mapsto(\gamma(t),\theta),\end{aligned}$$

then by (3.3) we calculate

$$\begin{aligned}
-\int_{h(\mathbb{T}^2)} f^*\omega_\varphi &= \int_{h(\mathbb{T}^2)} \iota_{Y_\theta}\omega \wedge d\theta \\
&= \int_0^1 \int_0^1 \iota_{Y_\theta}\omega \left[\frac{d\gamma}{dt}\right] dt\, d\theta \\
&= \int_0^1 \left(\int_0^1 \iota_{Y_\theta}\omega\, d\theta\right) \left[\frac{d\gamma}{dt}\right] dt \\
&= \langle Flux(\varphi), \overline{a}\rangle \\
&\neq 0.
\end{aligned}$$

This implies that $[h^* f^*\omega_\varphi] \neq 0$ in $H^2(\mathbb{T}^2, \mathbb{R})$. From Example 2.3 and Proposition 2.5, we have

$$u_{f^*\omega_\varphi}(s) \gtrsim u_{h^*f^*\omega_\varphi}(s) \sim s.$$

$\square$

Corollary 3.4. *Let (M, ω) be a symplectic manifold and $(M_\varphi, \omega_\varphi)$ be the mapping torus with the induced Hamiltonian structure in (3.6). If $Flux(\varphi) \neq 0$, then $(M_\varphi, \omega_\varphi)$ admits a stable Hamiltonian structure but no virtually contact structure.*

Proof. As mentioned in Remark 3.2 $\ker(f^*\omega_\varphi)$ is spanned by the vector field $\frac{\partial}{\partial\theta} - Y_\theta$. In order to define a stable structure on $(M_\varphi, \omega_\varphi) \cong (M \times \mathbb{S}^1, f^*\omega_\varphi)$, we choose the stabilizing 1-form λ in Definition 2.10 as $\pi_\theta^* d\theta$. Since $\lambda = \pi_\theta^* d\theta$ is closed, the first condition in (2.2) holds trivially. One can verify that $\lambda \wedge (f^*\omega_\varphi)^n = \pi_\theta^* d\theta \wedge \pi^*\omega^n$, $2n = \dim M$ and it is nowhere vanishing which implies that the second condition in (2.2) also holds true. Thus $(M_\varphi, \omega_\varphi)$ admits a stable Hamiltonian structure with a stabilizing 1-form $\pi_\theta^* d\theta$. By Proposition 3.3, the Hamiltonian structure ω_φ has a cofilling function of at least linear type. This means that there is no bounded primitive of ω_φ even in the universal cover and hence $(M_\varphi, \omega_\varphi)$ cannot be a virtually contact structure. $\square$

In order to obtain primitives of the Hamiltonian structures, we now consider the lifted structures on the universal covers. The lifted Hamiltonian structure $(\widetilde{M_{\mathbf{1}}}, \widetilde{\omega_{\mathbf{1}}})$ is clearly isomorphic to $(\widetilde{M} \times \mathbb{R}, \widetilde{\pi}^*\widetilde{\omega})$, where $\widetilde{\pi} : \widetilde{M} \times \mathbb{R} \to \widetilde{M} : (\widetilde{m}, r) \mapsto \widetilde{m}$. Note that $\widetilde{M_\varphi} = \widetilde{M} \times \mathbb{R}$ and $\widetilde{\omega}_\varphi = \widetilde{\pi}_\varphi^*\widetilde{\omega}$, where $\widetilde{\pi}_\varphi : \widetilde{M_\varphi} \to \widetilde{M}$ is the lift of $\pi_\varphi : M_\varphi \to M$. Since $\widetilde{\pi} = \widetilde{\pi}_\varphi$, we have $(\widetilde{M_\varphi}, \widetilde{\omega}_\varphi) \cong (\widetilde{M} \times \mathbb{R}, \widetilde{\pi}^*\widetilde{\omega})$.

Even though both lifted structures $(\widetilde{M_{\mathbf{1}}}, \widetilde{\omega_{\mathbf{1}}})$, $(\widetilde{M_\varphi}, \widetilde{\omega}_\varphi)$ are isomorphic to $(\widetilde{M} \times \mathbb{R}, \widetilde{\pi}^*\widetilde{\omega})$, they have different deck transformations as follows: An element $n \in \mathbb{Z} \cong \pi_1(\mathbb{S}^1) \hookrightarrow \pi_1(M_{\mathbf{1}})$ induces a translation $(\widetilde{m}, r) \mapsto (\widetilde{m}, r+n)$ on the universal cover $\widetilde{M_{\mathbf{1}}}$, while $n \in \mathbb{Z} \cong \pi_1(\mathbb{S}^1) \hookrightarrow \pi_1(M_\varphi)$ gives a twisted one $(\widetilde{m}, r) \mapsto (\widetilde{\varphi}^n(\widetilde{m}), r+n)$ on $\widetilde{M_\varphi}$. Here $\widetilde{\varphi} : \widetilde{M} \to \widetilde{M}$ is the lift of $\varphi : M \to M$ and $\widetilde{\varphi}^n$ is the n-th iterates of $\widetilde{\varphi}$.

We next consider the lift $\widetilde{f} : \widetilde{M_{\mathbf{1}}} \to \widetilde{M_\varphi}$ of $f : M_{\mathbf{1}} \to M_\varphi$. Since $\widetilde{M_{\mathbf{1}}} = \widetilde{M} \times \mathbb{R} = \widetilde{M_\varphi}$, it suffices to define the following:

$$\begin{aligned}
\widetilde{f} : \widetilde{M} \times \mathbb{R} &\to \widetilde{M} \times \mathbb{R} \\
(\widetilde{m}, r) &\mapsto (\widetilde{\varphi}_r(m), r).
\end{aligned}$$

Here $\widetilde{\varphi}_r = \widetilde{\varphi}_{r-\lfloor r \rfloor} \circ \widetilde{\varphi}^{\lfloor r \rfloor}$, where $\lfloor r \rfloor$ is the largest integer not greater than r and $\widetilde{\varphi}_\theta$ is the lift of φ_θ for $0 \leq \theta < 1$. We summarize the Hamiltonian structures, maps and its lifts into the

following diagram:

$$
\begin{tikzcd}
(\widetilde{M},\widetilde{\omega}) \arrow[d, "p"] & (\widetilde{M}_{\mathbf{1}},\widetilde{\omega}_{\mathbf{1}}) \arrow[l, "\widetilde{\pi}"'] \arrow[r, "\widetilde{f}"] \arrow[d, "p_{\mathbf{1}}"] & (\widetilde{M}_\varphi,\widetilde{\omega}_\varphi) \arrow[d, "p_\varphi"] \\
(M,\omega) & (M_{\mathbf{1}},\omega_{\mathbf{1}}) \arrow[l, "\pi"'] \arrow[r, "f"] & (M_\varphi,\omega_\varphi)
\end{tikzcd}
\tag{3.9}
$$

Here $p_{\mathbf{1}}$, p_φ are the natural projections.

Now we consider Riemannian metrics on the above spaces. Let g be a Riemannian metric on M, g_θ be the standard metric on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ which is induced by the Euclidean metric on $\mathbb{R}$. We consider a product Riemannian metric $g_{\mathbf{1}} := g \oplus g_\theta$ on $M_{\mathbf{1}} = M \times \mathbb{S}^1$ and $\widetilde{g}$, $\widetilde{g}_{\mathbf{1}}$ be lifts to the corresponding universal covers $\widetilde{M}$, $\widetilde{M} \times \mathbb{R}$ respectively.

The lifted Hamiltonian structure $(\widetilde{M}_\varphi, \widetilde{\omega}_\varphi) \cong (\widetilde{M} \times \mathbb{R}, \widetilde{\pi}^*\widetilde{\omega})$ admits a bounded primitive $\widetilde{\pi}^*\lambda \in \Omega^1(\widetilde{M}_\varphi)$, i.e. $d(\widetilde{\pi}^*\lambda) = \widetilde{\pi}^*\widetilde{\omega}$, with respect to $\widetilde{g}_{\mathbf{1}}$. Here the primitive 1-form $\lambda \in \Omega^1(\widetilde{M})$ exists and bounded with respect to $\widetilde{g}$, since our manifold (M,ω) is symplectically hyperbolic. This immediately implies that the pull-back $\widetilde{f}^*\widetilde{\omega}_\varphi$ also has a bounded primitive with respect to the pull-back metric $\widetilde{f}^*\widetilde{g}_{\mathbf{1}}$. Proposition 3.3 guarantees, however, that $\widetilde{f}^*\widetilde{\omega}_\varphi$ never admits a bounded primitive with respect to the metric $\widetilde{g}_{\mathbf{1}}$, when φ is a non-Hamiltonian symplectomorphism. Note especially that $\widetilde{f}^*\widetilde{g}_{\mathbf{1}}$ cannot be expressed as a lift of a Riemannian metric on $M_{\mathbf{1}}$.

We now investigate the pull-back metric $\widetilde{f}^*\widetilde{g}_{\mathbf{1}}$ on $\widetilde{M}_{\mathbf{1}}$. For $n \in \mathbb{Z} \subset \mathbb{R}$, $(\widetilde{m}, n) \in \widetilde{M}_{\mathbf{1}}$ and $(\widetilde{m}_i, 0) \in T_{(\widetilde{m},n)}\widetilde{M}_{\mathbf{1}}$, we have

$$
\begin{aligned}
(\widetilde{f}^*\widetilde{g}_{\mathbf{1}})_{(\widetilde{m},n)}\big((\widetilde{m}_1,0),(\widetilde{m}_2,0)\big) &= (\widetilde{g}_{\mathbf{1}})_{\widetilde{f}(\widetilde{m},n)}\big(\widetilde{f}_*(\widetilde{m}_1,0),\widetilde{f}_*(\widetilde{m}_2,0)\big) \\
&= (\widetilde{g}_{\mathbf{1}})_{\widetilde{f}(\widetilde{m},n)}\big((d\widetilde{\varphi}^n(\widetilde{m})[\widetilde{m}_1],0),(d\widetilde{\varphi}^n(\widetilde{m})[\widetilde{m}_2],0)\big) \\
&= \widetilde{g}_{\widetilde{\varphi}^n(\widetilde{m})}\big(d\widetilde{\varphi}^n(\widetilde{m})[\widetilde{m}_1], d\widetilde{\varphi}^n(\widetilde{m})[\widetilde{m}_2]\big) \\
&= g_{\varphi^n(m)}\big(d\varphi^n(m)[m_1], d\varphi^n(m)[m_2]\big),
\end{aligned}
\tag{3.10}
$$

where $m = p(\widetilde{m})$ and $m_i = p_*(\widetilde{m}_i)$.

4. Proof of Main Theorem

When φ is Hamiltonian diffeomorphism on (M,ω), as mentioned in Introduction, the Polterovich's result implies $\Gamma_n(\varphi) \gtrsim n$. So we only need to consider a symplectomorphism $\varphi \in Symp_0(M,\omega)$ with a non-vanishing flux.

We remark that we can choose a primitive $\widetilde{f}^*\widetilde{\pi}^*\lambda \in \Omega^1(\widetilde{M}_{\mathbf{1}})$ of $\widetilde{f}^*\widetilde{\omega}_\varphi$ which is bounded with respect to the twisted metric $\widetilde{f}^*\widetilde{g}_{\mathbf{1}}$. But $\widetilde{f}^*\widetilde{\pi}^*\lambda$ has at least linear growth with respect to the standard metric $\widetilde{g}_{\mathbf{1}}$ by Proposition 3.3. Now we interpret the difference between $\widetilde{g}_{\mathbf{1}}$ and $\widetilde{f}^*\widetilde{g}_{\mathbf{1}}$ into the growth rate of $\Gamma_n(\varphi)$.

Let us fix a primitive

$$
\widetilde{f}^*\widetilde{\pi}^*\lambda = \widetilde{\pi}^*\lambda + r \cdot \widetilde{\pi}^*(\iota_{\widetilde{Y}_r}\widetilde{\omega}) \tag{4.1}
$$

of $\widetilde{f}^*\widetilde{\omega}_\varphi$ in Lemma 3.1. Here r is the coordinate for $\mathbb{R} = \widetilde{\mathbb{S}}^1$ and $\widetilde{Y}_r$ is a vector field on $\widetilde{M}_{\mathbf{1}}$ which is the lift of Y_θ. Without loss of generality, we may assume that $[\iota_{Y_t}\omega] \in H^1(M,\mathbb{R})$ is non-trivial for $t = 0$ which implies that $\max_{z\in\widetilde{M}} |(\iota_{\widetilde{Y}_n}\widetilde{\omega})|_{\widetilde{g}}$ is positive for all $n \in \mathbb{N} \subset \mathbb{R}$. Now

we pick a point $m \in M$ such that

$$|(\iota_{Y_0}\omega)_m|_g = |(\iota_{\widetilde{Y}_n}\widetilde{\omega})_{\widetilde{m}}|_{\widetilde{g}} > 0, \quad \forall n \in \mathbb{N} \subset \mathbb{R}.$$

Since $|\widetilde{\pi}^*\lambda|_{\widetilde{g}_{\mathbf{1}}}$ is bounded,

$$|(\widetilde{f}^*\widetilde{\pi}^*\lambda)_{(\widetilde{m},n)}|_{\widetilde{g}_{\mathbf{1}}} \sim n \cdot |(\iota_{\widetilde{Y}_n}\widetilde{\omega})_{\widetilde{m}}|_{\widetilde{g}} \sim n, \quad \forall n \in \mathbb{N} \subset \mathbb{R}.$$

By the definition of the norm, there should be a sequence of tangent vectors $\{X_n\}_{n\in\mathbb{N}}$, $X_n \in T_{(\widetilde{m},n)}\widetilde{M}_{\mathbf{1}}$ which meets the following conditions:

- $\|X_n\|_{\widetilde{g}_{\mathbf{1}}} = \|\widetilde{\pi}_*X_n\|_{\widetilde{g}} = \|p_*\widetilde{\pi}_*X_n\|_g = 1, \quad \forall n \in \mathbb{N};$
- $(\widetilde{f}^*\widetilde{\pi}^*\lambda)_{(\widetilde{m},n)}(X_n) \sim n,$

where $\widetilde{\pi} : \widetilde{M}_{\mathbf{1}} \to \widetilde{M}$, $p : \widetilde{M} \to M$ are the same as in (3.9). Note that X_n has no r-direction component, because $\widetilde{f}^*\lambda_\varphi$ in (4.1) does not have dr-part.

Now we change the metric $\widetilde{g}_{\mathbf{1}}$ into $\widetilde{f}^*\widetilde{g}_{\mathbf{1}}$ on $\widetilde{M}_{\mathbf{1}}$ as follows:

$$\begin{aligned} n \sim & (\widetilde{f}^*\widetilde{\pi}^*\lambda)_{(\widetilde{m},n)}(X_n) \\ \leq & \sup_{z\in\widetilde{M}_{\mathbf{1}}} |(\widetilde{f}^*\widetilde{\pi}^*\lambda)_z|_{\widetilde{f}^*\widetilde{g}_{\mathbf{1}}} \cdot \|X_n\|_{\widetilde{f}^*\widetilde{g}_{\mathbf{1}}} \\ = & \sup_{z\in\widetilde{M}_\varphi} |(\widetilde{\pi}^*\lambda)_z|_{\widetilde{g}_{\mathbf{1}}} \cdot \|X_n\|_{\widetilde{f}^*\widetilde{g}_{\mathbf{1}}} \\ \leq & C \cdot \|X_n\|_{\widetilde{f}^*\widetilde{g}_{\mathbf{1}}}, \end{aligned}$$

where the constant $C > 0$ comes from the fact that $\widetilde{\pi}^*\lambda$ is bounded with respect to the metric $\widetilde{g}_{\mathbf{1}}$. Since the tangent vector X_n has no r-direction, we use (3.10) to obtain

$$\begin{aligned} n^2 \lesssim & \|X_n\|^2_{\widetilde{f}^*\widetilde{g}_{\mathbf{1}}} \\ = & (\widetilde{f}^*\widetilde{g}_{\mathbf{1}})_{(\widetilde{m},n)}(X_n, X_n) \\ = & g_{\varphi^n(m)}\big(d\varphi^n(m)[p_*\widetilde{\pi}_*X_n], d\varphi^n(m)[p_*\widetilde{\pi}_*X_n]\big) \\ \leq & \max_{m\in M} |d\varphi^n(m)|^2_g \cdot g(p_*\widetilde{\pi}_*X_n, p_*\widetilde{\pi}_*X_n) \\ = & \max_{m\in M} |d\varphi^n(m)|^2_g. \end{aligned}$$

Hence we conclude that $\max_{m\in M} |d\varphi^n(m)|_g$ has at least linear growth.

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